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On the rational derivation of a hierarchy of dynamic equations for a homogeneous, isotropic, elastic plate

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Abstract

Flexural equations of motion for a homogeneous, isotropic, elastic plate are derived by an antisymmetric expansion in the thickness coordinate of the displacement components. All but the lowest-order expansion functions are eliminated with the help of the three-dimensional equations of motion, and are plugged into the boundary conditions. Eliminating between these, an equation is obtained for the mean-plane vertical displacement which also includes arbitrary loading on the plate surface. This equation can be truncated to any order in the thickness and it is in particular noted that the corresponding dispersion relation seems to correspond to a power series expansion of the exact Rayleigh–Lamb dispersion relation to all orders. Various truncations of the equation are discussed and are compared numerically with each other, the exact three-dimensional solution and Mindlin's plate equation. Both the dispersion relation and the corresponding displacement components as well as an excitation problem are used for the comparisons. The theories are reasonably close to each other and in order to be on the safe side none of them should in fact be used for frequencies above the cutoff of the second flexural mode.

Keywords: Elastodynamics; Plates; Mindlin's equation; Kirchhoff's equation

1. Introduction

Vibrations and elastic wave propagation in homogeneous, isotropic, elastic plates have many important technical applications and have therefore been investigated extensively. The classical paper by Mindlin (1951) takes rotary inertia and shear deformation into account and also introduces the shear coefficient which has later been the cause of much debate. A satisfactory way to determine the shear coefficient has very recently been given by Stephen (1997) who makes a comparison with the low frequency series expansion of the exact Rayleigh–Lamb equation and matches the second-order terms. Many attempts at more refined theories have been made, but from a practical point of view it must be regarded as doubtful if these are successful. Recent publications with many references include Kaplunov et al. (1988), Stephen (1997),

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Muller and Touratier (1996), Jemielita (1990) and Ciarlet (1997). The static case is treated by Bisegna and Sacco (1997), Destuynder and Salaun (1996), Sutyrin and Hodges (1996), Mielke (1995), Rogers et al. (1992) and Reissner (1985).

The usual way of developing a plate theory is to start from a three-dimensional variational formulation and some more or less plausible kinematical assumptions. A more systematic approach is to develop some kind of series expansion technique and then retain as many terms as seems appropriate. Selezov (1994) uses series expansions for the displacements which are then plugged into the three-dimensional equation of motion. The resulting equations are not studied systematically and no numerical results are given. Selezov notes that only every second truncation leads to a hyperbolic system. Losin (1997) uses a much more involved procedure which in effect leads to a similar result. In particular, he derives a sixth-order hyperbolic equation for which the dispersion relation is investigated.

In the present paper, we start out in the same manner as Selezov (1994). Thus, we perform antisymmetric expansions in the thickness coordinate of the displacement components which are then inserted into the three-dimensional equations of motion. All but the three lowest-order unknowns are then eliminated with the help of the equations of motion, and it is noted that this elimination procedure does not involve the thickness of the plate. The boundary conditions then become series in the plate thickness which can be truncated in various ways. At the lowest-order, the Kirchhoff plate equation is recovered. At the next level, a Mindlin-type equation is obtained, but with somewhat different wave-speeds. Going one step further, the sixth-order equation of Losin (1997) is recovered. We also discuss the wave equation that is obtained for the antisymmetric SH waves.

Most earlier investigations of dynamic plate equations study the dispersion relation numerically as a means to determine the accuracy of the theories. Usually, comparisons are made with the exact antisymmetric Rayleigh–Lamb dispersion relation. This comparison is often extended to high frequencies and even to infinity, see e.g. Muller and Touratier (1996). It is, however, well known that the exact fundamental Rayleigh–Lamb wave-speed approaches the Rayleigh wave-speed and the corresponding mode approaches an antisymmetric pair of Rayleigh surface waves along the plate surfaces. No kinematical assumptions used in plate theories can match this mode behavior and it is therefore inappropriate to extend the comparison to high frequencies. To really investigate the useful range of frequencies, we study both the displacement field in the modes and an excitation problem with a harmonic pressure on a part of the plate. It will be seen that the studied plate theories can be safely used only for frequencies below the cutoff of the second flexural mode, i.e. when the plate thickness is at most half a shear wavelength.

2. Series expansions

Consider a plate of thickness $2h$ bounded by the two planes $z = \pm h$. The plate is homogeneous, isotropic and linearly elastic with density ρ and Lamé constants λ and μ. The three-dimensional equations of motion for the displacement field with components u,v,w are

$$(\lambda+\mu)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial^2 w}{\partial x\,\partial z}\right)+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\rho\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

$$(\lambda+\mu)\left(\frac{\partial^2 u}{\partial y\,\partial x}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y\,\partial z}\right)+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)=\rho\frac{\partial^2 v}{\partial t^2}, \tag{2}$$

$$(\lambda+\mu)\left(\frac{\partial^2 u}{\partial z\,\partial x}+\frac{\partial^2 v}{\partial z\,\partial y}+\frac{\partial^2 w}{\partial z^2}\right)+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)=\rho\frac{\partial^2 w}{\partial t^2}. \tag{3}$$

On the plate surfaces, the traction vector with components T_x, T_y, P should be specified. This yields the boundary conditions on $z = h$:

$$\sigma_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = T_x, \tag{4}$$

$$\sigma_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = T_y, \tag{5}$$

$$\sigma_{zz} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial w}{\partial z} = P \tag{6}$$

with similar boundary conditions on $z = -h$. In most cases, the traction vector, and in particular its tangential components, vanishes. For the sake of generality, we keep all three components.

We are only interested in flexural waves in the plate and to this end, we assume an antisymmetric solution. To comply with this, we also assume that the applied surface traction is antisymmetric. Then we only need the boundary conditions Eqs. (4)–(6) on the top surface. The displacement components are thus expanded as

$$u(x, y, z, t) = zu_1(x, y, t) + z^3u_3(x, y, t) + \cdots \tag{7}$$

$$v(x, y, z, t) = zv_1(x, y, t) + z^3v_3(x, y, t) + \cdots \tag{8}$$

$$w(x, y, z, t) = w_0(x, y, t) + z^2w_2(x, y, t) + \cdots \tag{9}$$

Inserting these expansions into Eqs. (1)–(3) and equating the coefficients of each power of z with zero individually, yields the following set of equations:

$$\mu\left(\frac{\partial^2 w_k}{\partial x^2} + \frac{\partial^2 w_k}{\partial y^2}\right) - \rho\frac{\partial^2 w_k}{\partial t^2} + (k+1)(\lambda+\mu)\left(\frac{\partial u_{k+1}}{\partial x} + \frac{\partial v_{k+1}}{\partial y}\right) + (k+1)(k+2)(\lambda+2\mu)w_{k+2} = 0, \quad k = 0, 2, \ldots, \tag{10}$$

$$(\lambda+2\mu)\frac{\partial^2 u_k}{\partial x^2} + \mu\frac{\partial^2 u_k}{\partial y^2} - \rho\frac{\partial^2 u_k}{\partial t^2} + (\lambda+\mu)\frac{\partial^2 v_k}{\partial x\,\partial y} + (k+1)(\lambda+\mu)\frac{\partial w_{k+1}}{\partial x} + (k+1)(k+2)\mu u_{k+2} = 0, \quad k = 1, 3, \ldots, \tag{11}$$

$$(\lambda+2\mu)\frac{\partial^2 v_k}{\partial y^2} + \mu\frac{\partial^2 v_k}{\partial x^2} - \rho\frac{\partial^2 v_k}{\partial t^2} + (\lambda+\mu)\frac{\partial^2 u_k}{\partial x\,\partial y} + (k+1)(\lambda+\mu)\frac{\partial w_{k+1}}{\partial y} + (k+1)(k+2)\mu v_{k+2} = 0, \qquad k = 1, 3, \ldots \tag{12}$$

Inserting the expansions into the boundary conditions Eqs. (4)–(6) gives

$$\frac{\partial w_0}{\partial x} + u_1 + h^2\left(\frac{\partial w_2}{\partial x} + 3u_3\right) + \cdots = \frac{T_x}{\mu}, \tag{13}$$

$$\frac{\partial w_0}{\partial y} + v_1 + h^2\left(\frac{\partial w_2}{\partial y} + 3v_3\right) + \cdots = \frac{T_y}{\mu}, \tag{14}$$

$$h\left(\lambda\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) + 2(\lambda + 2\mu)w_2\right) + h^3\left(\lambda\left(\frac{\partial u_3}{\partial x} + \frac{\partial v_3}{\partial y}\right) + 4(\lambda + 2\mu)w_4\right) + \cdots = P. \tag{15}$$

So far, we have followed exactly the same procedure as used by Selezov (1994). At first glance, the natural procedure – which is the one followed by Selezov – is to truncate the system by keeping terms up to a certain power of h in Eqs. (13)–(15) and all the corresponding expansion functions and the needed number of equations from Eqs. (10)–(12). The lowest meaningful order would seem to be the inclusion of h^2 terms and functions up to u_3 and v_3. This gives a nonhyperbolic system that does not resemble any of the usual plate equations. The next approximation also includes the h^3 term in Eq. (15) and therefore also the function w_4. The resulting system is hyperbolic and can be reduced to a single fourth-order differential equation that is of the same type as Mindlin's. The coefficients of this equation are different, however, and it is noted in particular that the coefficients are such that not even the correct form of Kirchhoff's equation is obtained in the low frequency limit.

3. Plate equations

The problem with a direct truncation of Eqs. (10)–(15) is in fact that when the truncated system is reduced to a single equation the coefficients in front of the various powers of h are not stable for increasing truncations. Noting that Eqs. (10)–(12) do not involve h at all, there is a very simple remedy for this unacceptable behavior. From Eq. (10) with $k = 0$, w_2 can be expressed in terms of w_0, u_1 and v_1. Then eliminating w_2 from Eqs. (11) and (12) with $k = 1$ gives u_3 and v_3 in terms of w_0, u_1 and v_1. This iterative process can be continued so that all the expansion functions can be expressed in terms of w_0, u_1 and v_1. It is to be noted that this process only involves simple algebraic manipulations that gives differential operators of increasing order for increasing orders of the expansion functions. We also stress that this elimination procedure does not involve the plate thickness h at all. Insertion into the boundary conditions (13)–(15) then yields three coupled equations involving w_0, u_1 and v_1 (the lengthy algebraic manipulations involved have been performed by means of Mathematica$^{\text{TM}}$)

$$\begin{aligned}
&\psi + \nabla w_0 + \frac{h^2}{2}\left[\frac{1}{c_s^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi - 2(1-\gamma)\nabla\nabla\cdot\psi - (1-2\gamma)\left(\frac{1}{c_s^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\nabla w_0\right] \\
&\quad + \frac{h^4}{24}\left[\frac{1}{c_s^4}\frac{\partial^4\psi}{\partial t^4} - \frac{2}{c_s^2}\left(\nabla^2 + (1-\gamma^2)\nabla\nabla\cdot\right)\frac{\partial^2\psi}{\partial t^2} + \nabla^4\psi + 4(1-\gamma)\nabla\nabla\cdot(\nabla^2\psi)\right. \\
&\quad \left. - \left(\frac{1-2\gamma^2}{c_s^4}\frac{\partial^4}{\partial t^4} - \frac{4-4\gamma-2\gamma^2}{c_s^2}\nabla^2\frac{\partial^2}{\partial t^2} + (3-4\gamma)\nabla^4\right)\nabla w_0\right] + \frac{h^6}{720}\left[\frac{1}{c_s^6}\frac{\partial^6\psi}{\partial t^6} - \frac{2(1-\gamma^3)}{c_s^4}\nabla\nabla\cdot\frac{\partial^4\psi}{\partial t^4}\right. \\
&\quad - \frac{3}{c_s^4}\nabla^2\frac{\partial^4\psi}{\partial t^4} + \frac{3}{c_s^2}\nabla^4\frac{\partial^2\psi}{\partial t^2} + \frac{6(1-\gamma^2)}{c_s^2}\nabla\nabla\cdot\nabla^2\frac{\partial^2\psi}{\partial t^2} - \nabla^6\psi - 6(1-\gamma)\nabla\nabla\cdot\nabla^4\psi + \left(-\frac{1-2\gamma^3}{c_s^6}\frac{\partial^6}{\partial t^6}\right. \\
&\quad \left.\left. + \frac{5-6\gamma^2-2\gamma^3}{c_s^4}\frac{\partial^4}{\partial t^4}\nabla^2 - \frac{9-6\gamma-6\gamma^2}{c_s^2}\nabla^4\frac{\partial^2}{\partial t^2} + (5-6\gamma)\nabla^6\right)\nabla w_0\right] + \mathrm{O}\left(h^8\right) = \frac{\boldsymbol{T}}{\mu},
\end{aligned} \tag{16}$$

$$h\left[-\nabla\cdot\psi+\frac{1}{c_s^2}\frac{\partial^2 w_0}{\partial t^2}-\nabla^2 w_0\right]+\frac{h^3}{6}\left[\left(-\frac{2-\gamma}{c_s^2}\frac{\partial^2}{\partial t^2}+(3-2\gamma)\nabla^2\right)\nabla\cdot\psi+\frac{\gamma}{c_s^4}\frac{\partial^4 w_0}{\partial t^4}\right.$$
$$\left.+\frac{1-3\gamma}{c_s^2}\nabla^2\frac{\partial^2 w_0}{\partial t^2}-(1-2\gamma)\nabla^4 w_0\right]+\frac{h^5}{120}\left[\left(-\frac{2-\gamma^2}{c_s^4}\frac{\partial^4}{\partial t^4}+\frac{6-2\gamma-2\gamma^2}{c_s^2}\nabla^2\frac{\partial^2}{\partial t^2}\right.\right.$$
$$\left.\left.-(5-4\gamma)\nabla^4\right)\nabla\cdot\psi+\frac{\gamma^2}{c_s^6}\frac{\partial^6 w_0}{\partial t^6}+\frac{2-2\gamma-3\gamma^2}{c_s^4}\nabla^2\frac{\partial^4 w_0}{\partial t^4}-\frac{5-6\gamma-2\gamma^2}{c_s^2}\nabla^4\frac{\partial^2 w_0}{\partial t^2}+(3-4\gamma)\nabla^6 w_0\right]$$
$$+\frac{h^7}{5040}\left[\left(-\frac{2-\gamma^3}{c_s^6}\frac{\partial^6}{\partial t^6}+\frac{8-3\gamma^2-2\gamma^3}{c_s^4}\nabla^2\frac{\partial^4}{\partial t^4}-\frac{12-3\gamma-6\gamma^2}{c_s^2}\nabla^4\frac{\partial^2}{\partial t^2}+(7-6\gamma)\nabla^6\right)\nabla\cdot\psi\right.$$
$$+\frac{\gamma^3}{c_s^8}\frac{\partial^8 w_0}{\partial t^8}+\frac{2-3\gamma^2-3\gamma^3}{c_s^6}\nabla^2\frac{\partial^6 w_0}{\partial t^6}-\frac{8-3\gamma-9\gamma^2-2\gamma^3}{c_s^4}\nabla^4\frac{\partial^4 w_0}{\partial t^4}+\frac{11-9\gamma-6\gamma^2}{c_s^2}\nabla^6\frac{\partial^2 w_0}{\partial t^2}$$
$$\left.-(5-6\gamma)\nabla^8 w_0\right]+O(h^9)=\frac{P}{\mu}. \tag{17}$$

Here, $c_s=\sqrt{\mu/\rho}$ is the transverse wavespeed and $\gamma=\mu/(\lambda+2\mu)$ is the squared quotient between the transverse and longitudinal wavespeeds. Two-dimensional vector notation in the x–y plane has been introduced so that

$$\psi=(u_1,v_1),\quad \nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right),\quad \nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \tag{18}$$

and of course $\nabla^4=(\nabla^2)^2$ and so on. In deriving Eqs. (16) and (17), expansion functions up to u_7, v_7 and w_8 have been used. It should be stressed that the coefficients in Eqs. (16) and (17) do not change if more expansion functions are included; this only affects the higher-order terms, which are not given.

Except that terms of order h^8 and higher have not been written down, no truncations have so far been employed. We can choose to truncate directly in Eqs. (16) and (17) or we can first eliminate ψ and obtain a single equation for w_0 which can then be truncated. Except for truncations where at most terms up to h^2 are kept, it is noted that the two truncation schemes are not equivalent.

The lowest meaningful truncation is to keep terms up to h^2 in Eqs. (16) and (17). The equations then resemble the corresponding Mindlin equations, with the h term in Eq. (17) being identical to the corresponding Mindlin equation except for the shear coefficient that multiplies the two Mindlin space derivative terms. The h independent terms in Eq. (16) are identical to the corresponding terms in Mindlin's equation, but the h^2 terms differ a bit, most notably the w_0 terms are completely missing in Mindlin's equation. Eliminating ψ a single equation for w_0 is obtained which has the same form as Mindlin's but with different coefficients. With this procedure not even the correct Kirchhoff equation is obtained in the low frequency limit. Within the present formulation it is, in fact, inconsistent to proceed in this way, because also the h^3 terms in Eq. (17) gives a contribution of the same order as the h^2 term in Eq. (16) when eliminating ψ (to see this, divide Eq. (17) with h throughout). However, using Eqs. (16) and (17) with terms up to h^3 gives a system that seems to be nonhyperbolic. Although the use of a nonhyperbolic system is not ruled out a priori (the Kirchhoff equation is parabolic), it is an undesirable feature which in fact unnecessarily limits the range of useful frequencies (we will comment more on this below when investigating various dispersion relations numerically).

With the above in mind, it is of course natural to eliminate ψ between Eqs. (16) and (17) before performing any truncations. Keeping terms up to h^6 the result is

$$\begin{aligned}
&\frac{1}{c_s^2}\frac{\partial^2 w_0}{\partial t^2} + \frac{h^2}{3}\left[4(1-\gamma)\nabla^4 w_0 - \frac{2(3-2\gamma)}{c_s^2}\nabla^2\frac{\partial^2 w_0}{\partial t^2} + \frac{3+\gamma}{2c_s^4}\frac{\partial^4 w_0}{\partial t^4}\right] + \frac{h^4}{15}\left[-4(1-\gamma)\nabla^6 w_0\right.\\
&\quad\left. + \frac{2(4-2\gamma-\gamma^2)}{c_s^2}\nabla^4\frac{\partial^2 w_0}{\partial t^2} - \frac{9+3\gamma-4\gamma^2}{2c_s^4}\nabla^2\frac{\partial^4 w_0}{\partial t^4} + \frac{5+10\gamma+\gamma^2}{8c_s^6}\frac{\partial^6 w_0}{\partial t^6}\right] + \frac{h^6}{720}\left[20(1-\gamma)\nabla^8 w_0\right.\\
&\quad - \frac{49-20\gamma-20\gamma^2}{c_s^2}\nabla^6\frac{\partial^2 w_0}{\partial t^2} + \frac{40+19\gamma-28\gamma^2-4\gamma^3}{c_s^4}\nabla^4\frac{\partial^4 w_0}{\partial t^4} - \frac{12+24\gamma-5\gamma^2-4\gamma^3}{c_s^6}\nabla^2\frac{\partial^6 w_0}{\partial t^6}\\
&\quad\left. + \frac{1+5\gamma+3\gamma^2}{c_s^8}\frac{\partial^8 w_0}{\partial t^8}\right] + \mathrm{O}(h^8)\\
&= \left(1 + \frac{h^2}{2}\left[-(3-2\gamma)\nabla^2 + \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\right] + \frac{h^4}{24}\left[(5-4\gamma)\nabla^4 - \frac{2(2-\gamma^2)}{c_s^2}\nabla^2\frac{\partial^2}{\partial t^2} + \frac{1}{c_s^4}\frac{\partial^4}{\partial t^4}\right]\right.\\
&\quad\left. + \frac{h^6}{720}\left[-(7-6\gamma)\nabla^6 + \frac{3(3-2\gamma^2)}{c_s^2}\nabla^4\frac{\partial^2}{\partial t^2} - \frac{5-2\gamma^3}{c_s^4}\nabla^2\frac{\partial^4}{\partial t^4} + \frac{1}{c_s^6}\frac{\partial^6}{\partial t^6}\right]\right)\frac{P}{\mu h}\\
&\quad + \left(1 + \frac{h^2}{6}\left[(3-2\gamma)\nabla^2 - \frac{2-\gamma}{c_s^2}\frac{\partial^2}{\partial t^2}\right] + \frac{h^4}{120}\left[(5-4\gamma)\nabla^4 - \frac{2(3-\gamma-\gamma^2)}{c_s^2}\nabla^2\frac{\partial^2}{\partial t^2} + \frac{2-\gamma^2}{c_s^4}\frac{\partial^4}{\partial t^4}\right]\right.\\
&\quad\left. + \frac{h^6}{5040}\left[-(7-6\gamma)\nabla^6 + \frac{3(4-\gamma-2\gamma^2)}{c_s^2}\nabla^4\frac{\partial^2}{\partial t^2} - \frac{8-3\gamma^2-2\gamma^3}{c_s^4}\nabla^2\frac{\partial^4}{\partial t^4} + \frac{2-\gamma^3}{c_s^6}\frac{\partial^6}{\partial t^6}\right]\right)\frac{\nabla\cdot\boldsymbol{T}}{\mu}.
\end{aligned} \tag{19}$$

Terms up to h^6 in Eq. (16) and h^7 in Eq. (17) have been used in deriving Eq. (19) and as for Eqs. (16) and (17), the given coefficients do not change if more expansion functions are included. It seems that all the operators appearing in Eq. (19) are hyperbolic (including those on the right-hand side) and the whole equation is thus hyperbolic at all truncations, which are performed so that all terms up to a certain power in h are retained.

In deriving Eq. (19) from Eqs. (16) and (17), the divergence is first taken of Eq. (16) and $\nabla\cdot\psi$ is then straightforwardly eliminated. In this process all solutions with $\nabla\cdot\psi = 0$ and $w_0 = 0$ are suppressed. These solutions are the antisymmetric SH waves in the plate. These solutions are often not discussed in plate theory, but see Stephen (1997) and Norris and Wang (1994). The equation of motion for the SH waves is easily obtained from Eq. (16) by introducing a scalar potential φ so that

$$\psi = \nabla\times(\boldsymbol{e}_z\varphi) = -\boldsymbol{e}_z\times\nabla\varphi, \tag{20}$$

where $\boldsymbol{e}_z$ is the unit vector normal to the plate. Using that $w_0 = 0$ and $\nabla\cdot\psi = 0$ for SH waves, Eq. (16) directly gives

$$\varphi + \frac{h^2}{2}\left[\frac{1}{c_s^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\varphi + \frac{h^4}{24}\left[\frac{1}{c_s^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]^2\varphi + \frac{h^6}{720}\left[\frac{1}{c_s^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]^3\varphi + \mathrm{O}(h^8) = T_s. \tag{21}$$

Here T_s is the transverse part of $\boldsymbol{T}$:

$$\boldsymbol{T} = \nabla T_p + \nabla\times(\boldsymbol{e}_z T_s), \tag{22}$$

which means that T_s can be obtained from the solution of the Poisson equation,

$$\nabla^2 T_s = \frac{\partial T_x}{\partial y} - \frac{\partial T_y}{\partial x}. \tag{23}$$

Alternatively, we may note that the SH part of ψ_x and ψ_y, i.e. u_1 and v_1, both satisfy Eq. (21) with right-hand sides T_x and T_y, respectively, with the extra condition that $\nabla \cdot \psi = 0$. If T_s is nonzero this is possibly a better approach to solve for the SH wave. It is noted that the terms in Eq. (21) correspond exactly to the first terms in a series expansion of the cosh function. Tentatively, the equation could be formally written

$$\cosh\left[h\left(\frac{1}{c_s^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)^{1/2}\right]\varphi = T_s. \tag{24}$$

Substituting $-\omega^2$ for $\partial^2/\partial t^2$ and $-k^2$ for ∇^2, the corresponding dispersion relation becomes

$$\cos\left[h\left(\frac{\omega^2}{c_s^2} - k^2\right)^{1/2}\right] = 0. \tag{25}$$

This is in fact the exact dispersion relation for the antisymmetric SH waves in the plate. The present method thus seems to yield some sort of asymptotic low frequency expansion of the exact three-dimensional theory.

4. Dispersion relations

To evaluate the various truncated forms of Eq. (19), we begin by comparing the corresponding dispersion relations both with exact three-dimensional theory (Achenbach, 1973, or Graff, 1975) and with Mindlin's theory with the shear coefficient chosen according to Stephen (1997) (but the exact value is not very important). To further illuminate the limitations of the theories, we also compute the displacement fields. In the next section, we make further comparisons for an excitation problem with a localized pressure.

Because of the problems with truncations of the pair Eqs. (16) and (17), we only discuss various truncations that may be performed on Eq. (19). The dispersion relations are obtained by substituting $-\omega^2$ for $\partial^2/\partial t^2$ and $-k^2$ for ∇^2 in Eq. (19). The following dispersion relations derived from Eq. (19) will be considered:

$$-k_s^2 + \tfrac{4}{3}(1-\gamma)k^4h^2 = 0, \tag{26}$$

$$-k_s^2 + \frac{h^2}{3}\left[4(1-\gamma)k^4 - 2(3-2\gamma)k^2k_s^2\right] + \frac{4}{15}(1-\gamma)k^6h^4 = 0, \tag{27}$$

$$-k_s^2 + \frac{h^2}{3}\left[4(1-\gamma)k^4 - 2(3-2\gamma)k^2k_s^2 + \frac{1}{2}(3+\gamma)k_s^4\right] = 0, \tag{28}$$

$$-k_s^2 + \frac{h^2}{3}\left[4(1-\gamma)k^4 - 2(3-2\gamma)k^2k_s^2 + \frac{1}{2}(3+\gamma)k_s^4\right] + \frac{h^4}{15}\left[4(1-\gamma)k^6 - 2\left(4-2\gamma-\gamma^2\right)k^4k_s^2 + \frac{1}{2}\left(9+3\gamma-4\gamma^2\right)k^2k_s^4 - \frac{1}{8}\left(5+10\gamma+\gamma^2\right)k_s^6\right] = 0. \tag{29}$$

Here, $k_s = \omega/c_s$ is the shear wave number. Eq. (26) is the classical Kirchhoff dispersion relation (written in an unorthodox way) which is obtained from the first two terms of Eq. (19). From this relation it is clear that $k_s = O(k^2)$ at low frequencies and the next level of approximation includes terms of order k^6 which leads to dispersion relation (27). The corresponding equation is nonhyperbolic exactly as the Kirchhoff equation. The lowest hyperbolic truncation of Eq. (19) leads to the dispersion relation (28), which is very similar to Mindlin's dispersion relation. The only difference is the somewhat different constants in the last two terms. It is to be noted that dispersion relation (28) is not a systematic low-frequency equation in that terms of order k^4, k^6 and k^8 are included but not all terms of order k^6 and k^8 are kept (this is of course also true for Mindlin's equation). The next level of hyperbolic approximation leads to dispersion relation (29). This is exactly the same dispersion relation that is derived by Losin (1997) in a somewhat more complicated way. Losin's plate equation thus includes terms up to h^4 in Eq. (19), but it should be noted that he excludes the traction vector on the plate surface and Losin's right-hand side therefore vanishes. (Losin plots the solutions of his dispersion relation, but does not include any other solutions, not even the exact one. He correctly claims that his solutions are close to the exact ones, but incorrectly claims that his fundamental phase velocity approaches the Rayleigh wavespeed at high frequencies. However, the discrepancy is very small, see our dispersion relation (29), in Fig. 1). Besides dispersion relations (26)–(29), we also consider Mindlin's dispersion relation and the exact antisymmetric Rayleigh–Lamb dispersion relation:

$$-\frac{3k_s^2}{4(1-\gamma)} + h^2\left(k^2 - \frac{\omega^2}{k'c_s^2}\right)\left(k^2 - \frac{\omega^2}{c_r^2}\right) = 0, \tag{30}$$

$$4k^2 h_s h_p \tan h_s h + \left(h_s^2 - k^2\right)^2 \tan h_p h = 0. \tag{31}$$

Here $c_r = \sqrt{E/\rho(1-\nu^2)}$, $h_s = \sqrt{k_s^2 - k^2}$, $h_p = \sqrt{k_p^2 - k^2}$, $k_p = \omega\sqrt{\rho/(\lambda + 2\mu)}$ is the longitudinal wavenumber and k' is the shear coefficient. We here choose k' in the best way according to Stephen (1997) (meaning that the asymptotic low frequency fit of the dispersion relation is optimal):

$$k' = \frac{5}{6-\nu}, \tag{32}$$

where ν is the Poisson ratio. It is very interesting to note that a low frequency series expansion of the exact dispersion relation (31) seems to give the dispersion relation from Eq. (19) to all orders (but it has only been checked for the terms given in Eq. (19)). From the manner in which Eq. (19) was derived, this is really to be expected. The series expansion of the exact dispersion relation is of course a different, although ad hoc, way of deriving Eq. (19) (without the right-hand side). One way to compare dispersion relations (28) (which is of the same form as Mindlin's) and (30) (Mindlin's) is to calculate the two wavespeeds in the principal parts. For $\nu = 0.25$ the principal parts of dispersion relation (28) gives the two wavespeeds $0.89c_s$ and $1.41c_s$ while the principal part of dispersion relation (30) gives the two wavespeeds $0.93c_s$ and $1.63c_s$. These are reasonably close although the higher speeds differ by about 15%. These differences in wavespeeds give differences also in the displacements that can sometimes be appreciable as will shortly be seen. Fig. 1 shows the solutions of dispersion relations (26)–(31) up to the wave number $kh = 5$. For dispersion relations (28)–(31), two solutions are plotted. The exact dispersion relation (31) has of course infinitely many solutions, relations (28) and (30) have two solutions and relation (29) has three solutions. The third solution to dispersion relation (29) is not given although it would fit into the plot (as would the third solution to the exact dispersion relation). The second solution to dispersion relation (29) becomes complex for $0.7 < kh < 2.3$ (and is then the complex conjugate of the third solution) but only the real part is plotted in Fig. 1. The Poisson ratio for the plots is $\nu = 0.25$. From Fig. 1, it is clear that the Kirchhoff dispersion relation is only valid for $k_s h < 0.3$, i.e only at very low frequencies. The nonhyperbolic dispersion relation (27) has an equally good low frequency asymptote as dispersion relations (29) and (30) (due to the optimal choice of k' in the Mindlin theory) and a better one than dispersion relation (28), but still it is only valid for at most $k_s h < 1$. The hyperbolic theories 3–5 all show a more or less reasonable agreement with the exact result in the plotted interval (and also for higher frequencies). For the fundamental solution, the agreement is in fact excellent, particularly for dispersion relation (29). That the hyperbolic theories seem superior is probably due to the fact that the exact three-dimensional theory is also hyperbolic.

To investigate the useful frequency ranges of the hyperbolic theories it is clearly not enough to investigate the dispersion relations, so to this end we now proceed to investigate the displacement fields of the modes for the various theories. Figs. 2 and 3 show the displacement components w and u, respectively, of

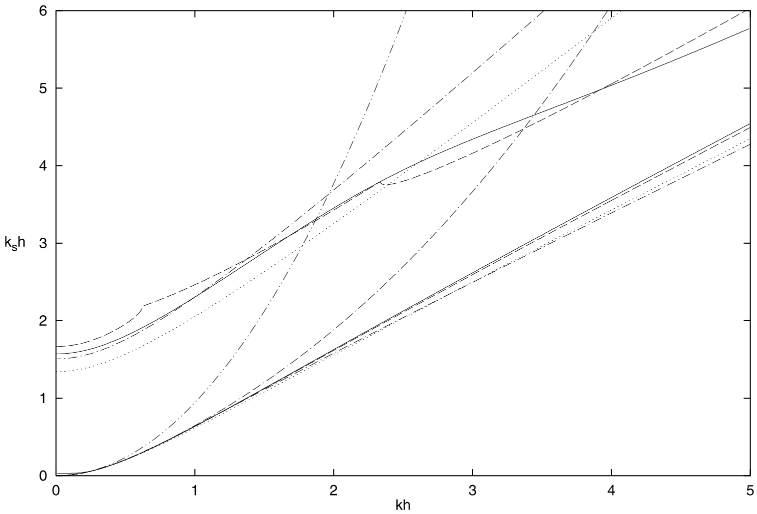

Fig. 1. The dispersion relations (wave numbers as functions of frequency) for up to two modes: (—) exact theory, (– · –) Mindlin's theory, (· · ·) second order hyperbolic theory, (– –) fourth order hyperbolic theory, (— · ·—) k^4 nonhyperbolic theory and (— — · — —) k^6 nonhyperbolic theory.

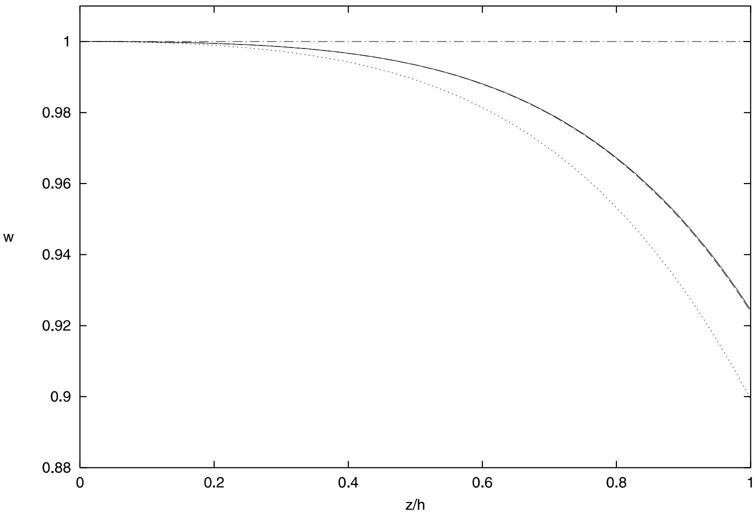

Fig. 2. The normal displacement w of the fundamental mode for the frequency $k_s h = 1$: (—) exact theory, (— · —) Mindlin's theory, (· · ·) second order hyperbolic theory and (— — —) fourth order hyperbolic theory.

the fundamental mode at frequency $k_s h = 1$ for theories 3–6. The displacements are given as a function of the normalized thickness coordinate z/h. The normalization is such that $w = 1$ at $z/h = 0$. It is noted that w is almost constant for all the theories. Theory 4 including terms up to h^4 is very close to the exact theory. Figs. 4 and 5 are corresponding plots at the higher frequency $k_s h = 2$. The agreement between the exact

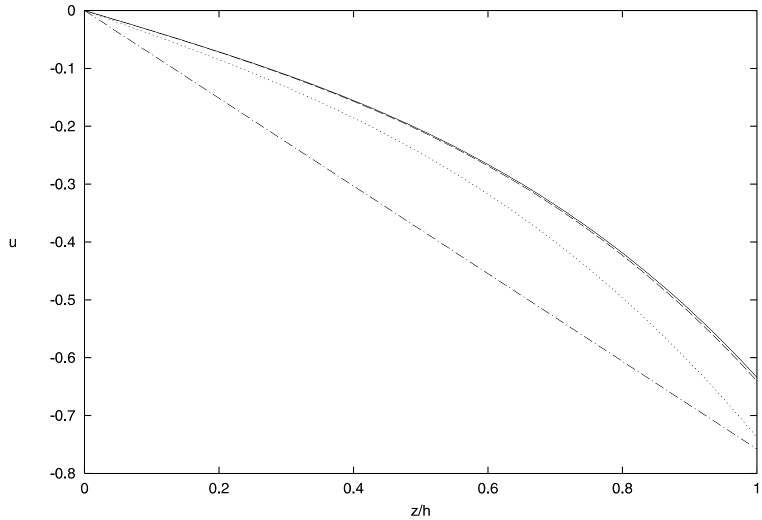

Fig. 3. Same as in Fig. 2 but for tangential displacement u.

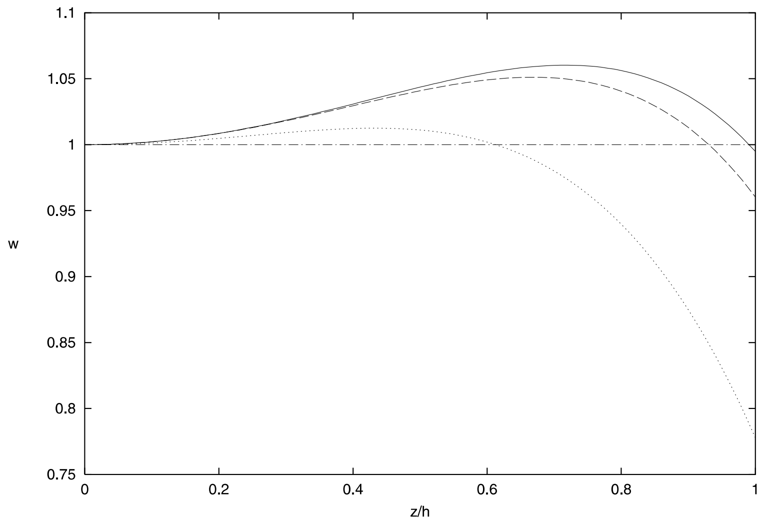

Fig. 4. Same as in Fig. 2 but for frequency $k_s h = 2$.

curve and the approximate ones is still reasonable with the h^4 curve being quite close. Figs. 6 and 7 show corresponding waves for the second mode, still at the frequency $k_s h = 2$. As u is greater than w in this mode the normalization is chosen so that all the theories have the same slope at $z/h = 0$ and so that $u = 1$ at $z/h = 1$ for the exact wave. The agreement between the waves is then good for u but for w, the solution is less satisfactory.

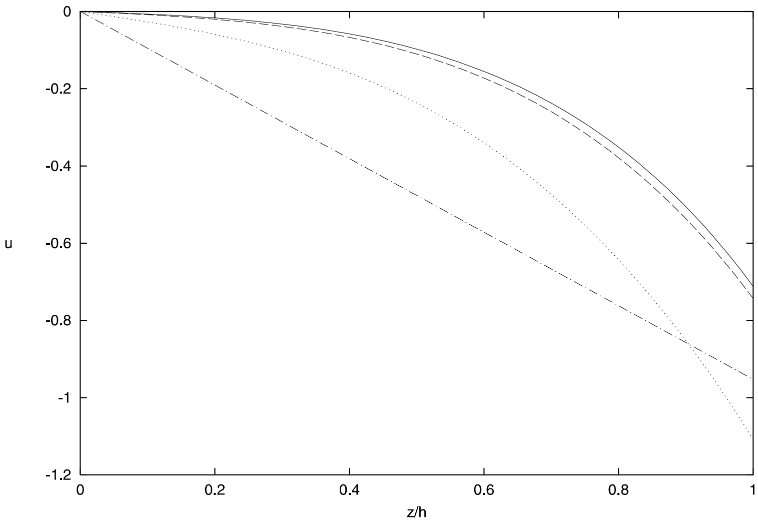

Fig. 5. Same as in Fig. 4 but for tangential displacement u.

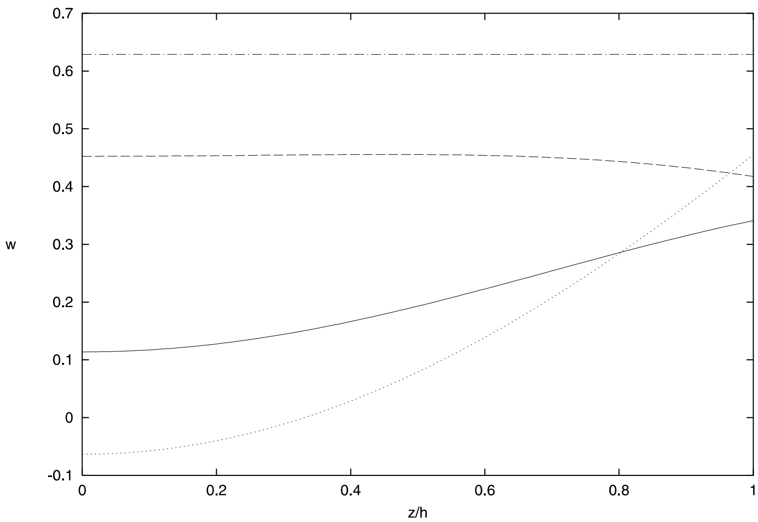

Fig. 6. Same as in Fig. 2 but for second mode and frequency $k_s h = 2$.

5. An excitation problem

To further evaluate the plate equations, in this section we solve a two-dimensional problem with an infinite plate subjected to a localized time-harmonic pressure. Thus, consider a plate of thickness $2h$ where the boundary condition on $z = h$ is

$$\sigma_{zz} = \begin{cases} P_0, & |x| < a, \\ 0, & |x| > a, \end{cases} \tag{33}$$

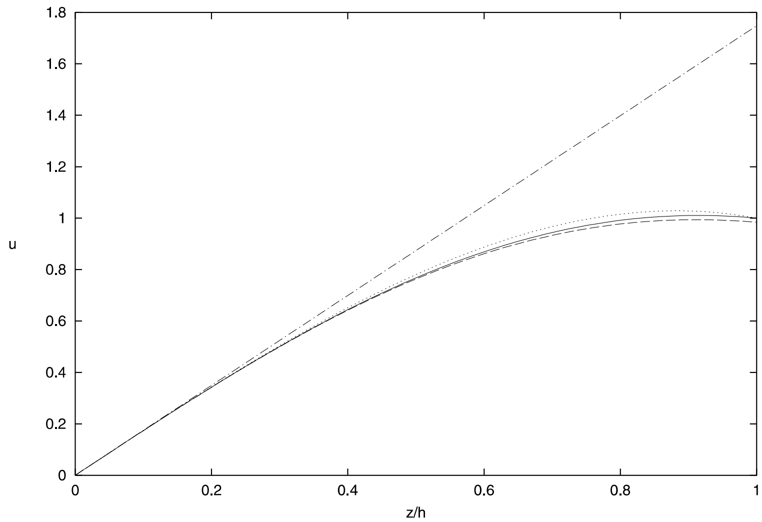

Fig. 7. Same as in Fig. 6 but for tangential displacement u.

$$\sigma_{zx} = \sigma_{zy} = 0. \tag{34}$$

An antisymmetric excitation is assumed so that $\sigma_{zz} = -P_0$, $|x| < a$, on the lower plate surface.

The time factor $\exp(-\mathrm{i}\omega t)$, where ω is the angular frequency, is suppressed throughout. As the problem is y independent, we easily obtain a solution in three-dimensional elasticity by applying a Fourier transform in x. The solution for the vertical displacement on the plate's mid-plane becomes

$$w = -\frac{2P_0 h}{\pi\mu}\int_0^\infty \frac{h_p\left[\left(k_s^2 - 2k^2\right)\cos(h_s h) + 2k^2\cos(h_p h)\right]\sin ka}{ka\left[\left(k_s^2 - 2k^2\right)^2\sin(h_p h)\cos(h_s h) + 4k^2 h_p h_s\cos(h_p h)\sin(h_s h)\right]}\cos(kx)\,\mathrm{d}k, \tag{35}$$

where h_p and h_s are defined below Eq. (31). The denominator in Eq. (35) is of course, recognized as the antisymmetric Rayleigh–Lamb dispersion relation (31). The half-infinite integral in Eq. (35) has to be computed numerically. As the integrand has at least one pole on the real axis, we have deformed the integration contour into the fourth quadrant of the complex k plane.

The Mindlin equation including the forcing term is

$$\left(\nabla^2 - \frac{1}{k'c_s^2}\frac{\partial^2}{\partial t^2}\right)\left(\nabla^2 - \frac{1}{c_r^2}\frac{\partial^2}{\partial t^2}\right)w + \frac{3}{c_r^2 h^2}\frac{\partial^2 w}{\partial t^2} = -\left(\frac{3(1-\nu)}{4} - \frac{1}{2k'h^2}\left(\nabla^2 - \frac{1}{c_r^2}\frac{\partial^2}{\partial t^2}\right)\right)\frac{qh}{\mu}. \tag{36}$$

For a y independent problem with fixed frequency and with $q = 2P_0$, $|x| < a$, this equation can be solved exactly. We have, however, chosen to solve it in the same way as above with a Fourier transform in x with the result that

$$w = -\frac{2P_0 h}{\pi\mu}\int_0^\infty \frac{3(1-\nu)/2h^2 + \frac{1}{k'}\left(k^2 - \omega^2/c_r^2\right)}{h^2(k^2 - \omega^2/c_r^2)(k^2 - \omega^2/k'c_s^2) - 3\omega^2/c_p^2}\,\frac{\sin(ka)}{k}\cos(kx)\,\mathrm{d}k. \tag{37}$$

The integral is evaluated in the same manner as the one in Eq. (35). For the various truncations of Eq. (19), the same procedure as for Mindlin's equation is employed. In this section, we only consider the two hyperbolic theories which are obtained by keeping all terms up to h^2 and up to h^4, respectively, including the right-hand side. The solutions are not given here as they have only obvious changes from the Mindlin solution (37). For the numerical result, we choose the pressure width equal to the plate thickness so that

$a = h$. The dimensionless vertical mid-plane displacement $\mu w/2P_0h$ is plotted as a function of distance along the plate x/h. The Poisson ratio is $v = 0.25$.

Fig. 8 shows the displacements from the exact three-dimensional theory, Mindlin's equation and the h^2 and h^4 hyperbolic equations. The frequency is $k_sh = 1$, i.e the plate thickness is about a third of the shear wavelength. All the theories agree well although the h^2 equation clearly is inferior both in amplitude and phase, the latter being somewhat surprising as the dispersion relation agrees very well in Fig. 1. Fig. 9 shows similar curves for the doubled frequency $k_sh = 2$. This is above the cutoff of the second plate mode for all the theories. Mindlin's equation still performs well, the h^2 equation has a small phase error and the h^4

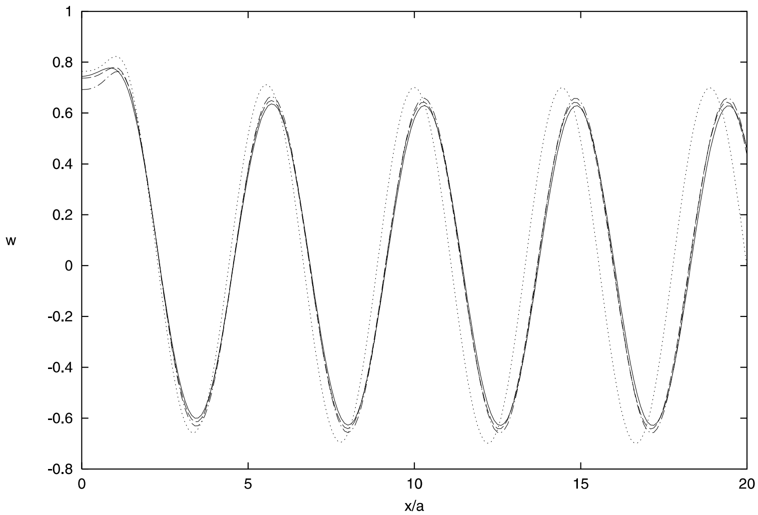

Fig. 8. Displacement w as function of distance x/h along the plate for the frequency $k_sh = 1$: (—) exact theory, (– · –) Mindlin's theory, (···) second order hyperbolic theory and (–) fourth order hyperbolic theory.

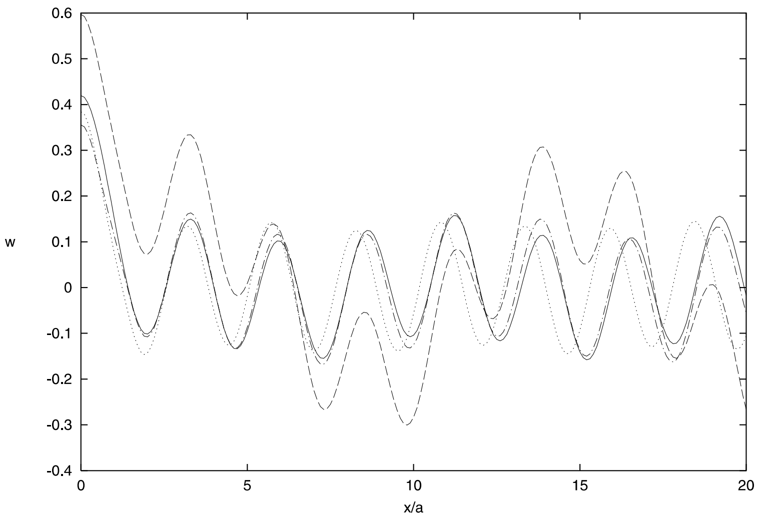

Fig. 9. Same as in Fig. 8 but with the frequency $k_sh = 2$.

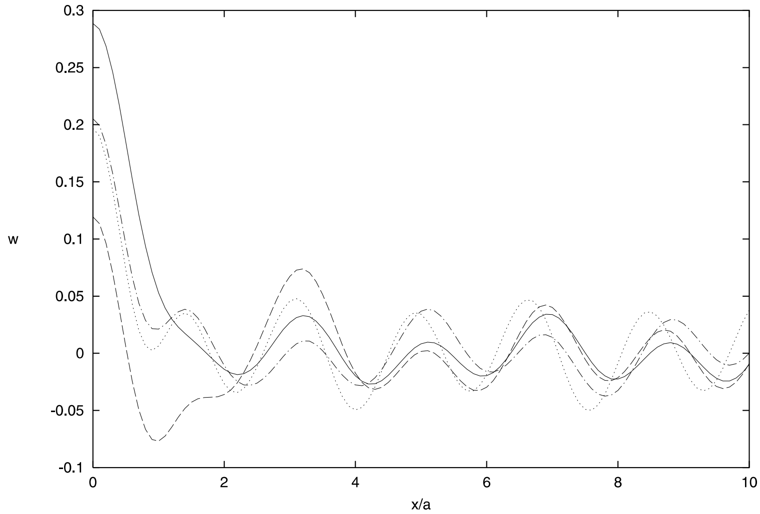

Fig. 10. Same as in Fig. 8 but with the frequency $k_s h = 3$.

equation has a too high amplitude in the second mode. The latter behavior may have something to do with the poor agreement of the h^4 dispersion relation for the second mode around $k_s h = 2$.

Finally, in Fig. 10, the frequency is increased to $k_s h = 3$. None of the approximate equations agree satisfactorily with the exact theory, particularly for $x/h < 2$. For $x/h > 4$, the h^4 equation gives the best fit and in fact agrees reasonably well with the exact theory.

6. Concluding remarks

We have investigated the derivation of dynamic plate equations by expanding the displacement components in power series in the thickness coordinate. The three-dimensional equations of motion then gives a hierarchy of equations where all but the unknowns of lowest-order are easily eliminated. The remaining equations (emanating from the three-dimensional boundary conditions) are three coupled equations which are power series in the plate thickness with derivative operators of corresponding orders. These equations can be truncated to any desired order, but a better approach seems to be to eliminate between the equations and obtain two equations in a single unknown each. One of these is an equation for the vertical mid-plane displacement for the flexural waves, the other is an equation for the antisymmetric SH waves in the plate. Forcing terms from (antisymmetric) surface tractions (both pressure and other forces) are included in the equations. The flexural and SH waves propagate independently of each other but in general they couple to each other at the edges of the plate (in the present paper we do not investigate the boundary conditions at plate edges, although this can be a delicate problem with boundary layer waves, etc.). We note that the dispersion relation for the flexural waves seems to agree to all orders with the low frequency expansion of the exact antisymmetric Rayleigh–Lamb dispersion relation. We also note that a systematic low frequency expansion including all terms to a certain order in k leads to a nonhyperbolic equation (like Kirchhoff's). To be useful at somewhat higher frequencies it seems, however, that it is best to use a hyperbolic equation (like Mindlin's) although this from a systematic point of view seems less satisfactory.

Numerical results are given for the dispersion relation, the displacements in the modes and for an excitation problem with a localized pressure. The exact three-dimensional theory, Mindlin's equation and some of the presented equations – particularly the hyperbolic equations of orders h^2 and h^4 – are compared with each other. The various equations all have their strengths and weaknesses and none of them can be claimed to be the 'best'. All give reasonable results for frequencies below the cutoff of the second mode (for shear wavelengths longer than twice the plate thickness). For frequencies above this cutoff, the result of the approximate theories are more inaccurate and when the shear wavelength is about the plate thickness the approximate theories should definitely be avoided.

The validity of the second mode of Mindlin's equation has been debated, see Stephen (1997). He calls the behavior of this mode 'schizophrenic' and states that it 'should be regarded as the inevitable, but meaningless, consequence of an otherwise remarkable approximate engineering theory'. This conclusion is drawn from the dispersion curve for the second mode when all frequencies are considered. We do not agree with Stephen's conclusion, because Mindlin's equation is only valid at low frequencies and the dispersion relations should only be compared for such frequencies. Also, Mindlin's equation predicts a reasonable mode shape (Figs. 6 and 7) and excitation (Fig. 9) at $k_s h = 2$. Close to the cutoff Mindlin's equation thus gives a fair description of the second mode.

The methodology used here could of course be applied to other similar problems for beams, plates and shells. Some such situations are presently being investigated.

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